

ON THE BETTER BEHAVED VERSION OF THE GKZ HYPERGEOMETRIC SYSTEM

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ABSTRACT. We consider a version of the generalized hypergeometric system introduced by Gelfand, Kapranov and Zelevinski (GKZ) suited for the case when the underlying lattice is replaced by a finitely generated abelian group. In contrast to the usual GKZ hypergeometric system, the rank of the better behaved GKZ hypergeometric system is always the expected one. We give largely self-contained proofs of many properties of this system. The discussion is intimately related to the study of the variations of Hodge structures of hypersurfaces in algebraic tori.

1. INTRODUCTION

The seminal paper of Gelfand, Kapranov and Zelevinsky [GKZ] introduced generalizations of the classical hypergeometric function as solutions to certain systems of partial differential equations defined by combinatorial data. These GKZ hypergeometric functions have been studied extensively in the subsequent years. In particular, they appear naturally in the study of mirror symmetry for hypersurfaces and complete intersections in toric varieties, see [BvS, HLY, S]. Unfortunately, the version of the GKZ hypergeometric system that is most suitable to mirror symmetry applications has received scant attention in the more general works on the GKZ systems, such as [MMW, SST]. The disconnect between the general theory and the most interesting (from our viewpoint) applications makes the theory seem far more formidable than it really is and inhibits the development of mirror symmetry.

In our paper we aim to bridge this gap by considering the *better behaved version* of the GKZ hypergeometric system. We deliberately try to make our arguments as self-contained as possible. In order to make the subject more accessible, we avoid any D -module technology. In the process we give a more algebraic treatment of the cohomology spaces of nondegenerate hypersurfaces in algebraic tori that were originally studied by Batyrev in [B].

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Let us first recall the original definition of GKZ hypergeometric system. Let $\mathcal{A} = \{v_1, \dots, v_n\}$ be a set of vectors in the lattice $N \cong \mathbb{Z}^d$ such that the elements of \mathcal{A} generate the lattice as an abelian group, and there exists a group homomorphism $\deg : N \rightarrow \mathbb{Z}$ such that $\deg(v) = 1$ for any element $v \in \mathcal{A}$. Let $L \subset \mathbb{Z}^n$ denote the lattice of integral relations among the elements of \mathcal{A} consisting of vectors $l = (l_i) \in \mathbb{Z}^n$ such that $l_1 v_1 + \dots + l_n v_n = 0$.

For any parameter $\beta \in N \otimes \mathbb{C}$, Gelfand, Kapranov and Zelevinsky [GKZ] considered a system of differential equations on the function $\Phi(x)$, $x = (x_1, \dots, x_n) \in \mathbb{C}^n$, consisting of the binomial equations

$$\left(\prod_{i, l_i > 0} (\partial_i)^{l_i} - \prod_{i, l_i < 0} (\partial_i)^{-l_i} \right) \Phi = 0, \quad l \in L,$$

and the linear equations

$$\left(\sum_{i=1}^n \mu(v_i) x_i \partial_i \right) \Phi = \mu(\beta) \Phi, \quad \text{for all } \mu \in M = \text{Hom}(N, \mathbb{Z}).$$

Gelfand, Kapranov and Zelevinsky showed that this system is holonomic, so the number of linearly independent solutions at a generic point is finite. Following Batyrev's observation [B, Section 14] that the periods of a Calabi–Yau hypersurface in a projective toric variety satisfy a GKZ system, its study gained further prominence in connection with mirror symmetry phenomena and algebra geometric applications.

The rank of the GKZ system (the dimension of its solution set at a generic point) and the solution set itself have also been the subject of numerous studies. Its expected dimension is equal to the normalized volume of the convex hull Δ of the elements of the set \mathcal{A} . However, unless the toric ideal associated to \mathcal{A} is Cohen–Macaulay, there are non-generic values of β for which the rank jumps. This rank discrepancy has been thoroughly investigated by many authors (see, for example, Adolphson [A], Saito, Sturmfels and Takayama [SST], Cattani, Dickenstein and Sturmfels [CDS]) and a quite definitive explanation for it has been obtained in the work of Matusevich, Miller and Walther [MMW].

In the present work, we focus on a *better behaved* version of the GKZ system whose space of solutions always has the expected dimension. We frame the definition in a context where the lattice is replaced by a finitely generated abelian group N , and the set \mathcal{A} is replaced by an n -tuple $\mathcal{A} = (v_1, \dots, v_n)$ of elements of N , with possible repetitions. Given a parameter β in $N \otimes \mathbb{C}$, the better behaved GKZ system consists of the equations

$$\partial_i \Phi_c = \Phi_{c+v_i}, \quad \text{for all } c \in K, \quad i \in \{1, \dots, n\}$$

and the linear equations

$$\sum_{i=1}^n \mu(v_i) x_i \partial_i \Phi_c = \mu(\beta - c) \Phi_c, \text{ for all } \mu \in M, c \in K.$$

A solution to the better behaved GKZ system is then a sequence of functions of n variables $(\Phi_c(x_1, \dots, x_n))_{c \in K}$, where K is the preimage under the map $N \rightarrow N \otimes \mathbb{R}$ of the cone $K_{\mathbb{R}}$ generated by the images of the elements v_i in $N \otimes \mathbb{R}$.

When N is a lattice and \mathcal{A} is a finite subset subject to the hyperplane condition, the better behaved GKZ equations on Φ_0 imply the usual GKZ equations on that function. The generalization presented in our work fits in the general context of ideas where the usual combinatorial framework of toric geometry is extended from toric varieties and their fans to that of toric Deligne-Mumford stacks and stacky fans provided in the work of Borisov, Chen and Smith [BoCS].

We now briefly discuss the contents of this paper. In section 2, we give the precise definition of the better behaved GKZ system. In section 3, we relate the spaces of solutions to the logarithmic Jacobian rings (Definition 3.3). In particular, we prove that the spaces of solutions have indeed the expected dimension, namely the product of the normalized volume of the polytope Δ and the torsion order of the abelian group N . In section 4, we investigate the effects of torsion in N and repetitions among the elements v_i . In section 5 we study the restriction map from the solution space of GKZ for the cone to that for its interior. This restriction map is familiar in mirror symmetry. For example, in the case of the quintic hypersurface in \mathbb{P}^4 , it amounts to considering the restriction map from the space of periods associated to the mirror Landau–Ginzburg model of \mathbb{P}^4 to the periods of the Calabi–Yau mirror quintic. The results of sections 5 are intimately related to the work of Batyrev [B, section 8], but our treatment is more algebraic and self-contained. In the last section, we discuss some variations of the main construction, as well as some open problems related to it.

Acknowledgements. Upon learning about our construction, in a letter to one of the authors, Alan Adolphson [A1] wrote us that he obtained a similar definition for a generalization of the GKZ system in the case of a lattice N . Hiroshi Iritani informed us that in a recent preprint [I], the better behaved GKZ system appears as a natural ingredient in his study of the quantum D -module associated to a toric complete intersection and the periods of its mirror. We would also like to thank Vladimir Retakh for a useful reference.

2. THE USUAL AND THE BETTER-BEHAVED VERSIONS OF THE GKZ HYPERGEOMETRIC SYSTEM

Throughout this paper, we will use the following notations. We are given a finitely generated abelian group N , and an n -tuple $\mathcal{A} = (v_1, \dots, v_n)$ of elements of N . We will denote by M the free abelian group $\text{Hom}(N, \mathbb{Z})$. We will assume that there exists an element $\deg \in M$ such that $\deg(v_i) = 1$ for all i . We will denote by Δ the convex hull of the set of v_i in $N \otimes \mathbb{R}$ and by $K_{\mathbb{R}}$ the cone $\mathbb{R}_{\geq 0}\Delta$. We will denote by K the preimage of $K_{\mathbb{R}}$ in N under the natural map $\pi : N \rightarrow N \otimes \mathbb{R}$ and by $[c]$, for $c \in K$, the corresponding elements in the semigroup ring $\mathbb{C}[K]$ or its variants that will be used below. We will further assume that $\pi(v_i)$ span the lattice $\pi(N)$ as a group. The finite abelian group $\text{tors}(N)$ is the torsion part of N and $|\text{tors}(N)|$ its order. For N torsion-free, we set $|\text{tors}(N)| = 1$.

The version of the GKZ hypergeometric system associated to a fixed parameter $\beta \in N \otimes \mathbb{C}$ which will be the central object of study of this paper is then defined as follows:

Definition 2.1. Consider the following system of partial differential equations on sequences of functions of n variables $(\Phi_c(x_1, \dots, x_n))_{c \in K}$:

$$(1) \quad \partial_i \Phi_c = \Phi_{c+v_i}, \quad \text{for all } c \in K, \ i \in \{1, \dots, n\}$$

$$(2) \quad \sum_{i=1}^n \mu(v_i) x_i \partial_i \Phi_c = \mu(\beta - c) \Phi_c, \quad \text{for all } \mu \in M, \ c \in K.$$

We will call this system the *better behaved GKZ* and will denote it by $\text{GKZ}(\mathcal{A}, K; \beta)$.

In order to simplify our notation, we will denote a solution to the better behaved GKZ by $\Phi_K(x_1, \dots, x_n)$. Alternatively, it can be viewed as a function in n variables

$$\Phi_K(x_1, \dots, x_n) = \sum_{c \in K} \Phi_c(x_1, \dots, x_n) [c]$$

with values in the completion $\mathbb{C}[K]^c$ of the ring $\mathbb{C}[K]$.

Remark 2.2. It is clear that one can reformulate the above system as a system of PDEs on a finite collection of functions of (x_1, \dots, x_n) . Indeed, the set K_{prim} of elements $v \in K$ such that $v - v_i \notin K$ for all i is finite. The functions Φ_c for $c \in K_{\text{prim}}$ then determine the rest of Φ_c . In fact, the number of PDEs can also be made finite, in view of the following. The relations (2) for $c \in K$, together with the relation

(1) implies the relations (2) for $c + v_i$. Consequently, one only needs to use (2) for $c \in K_{\text{prim}}$. The relations (1) can then be restated as

$$(3) \quad \left(\prod_{i=1}^n \partial_i^{k_i} \right) \Phi_{c_1} = \left(\prod_{i=1}^n \partial_i^{l_i} \right) \Phi_{c_2}$$

for all $k_i, l_i \in \mathbb{Z}_{\geq 0}$ such that

$$c_1 + \sum_i k_i v_i = c_2 + \sum_i l_i v_i$$

and $c_1, c_2 \in K_{\text{prim}}$. To see that (3) follows from a finite number of relations of this type, note that they correspond to the \mathbb{C} -basis of the module over the polynomial ring $\mathbb{C}[\partial_1, \dots, \partial_n]$ which is the kernel of the natural map

$$\mathbb{C}[K_{\text{prim}}] \otimes \mathbb{C}[\partial_1, \dots, \partial_n] \rightarrow \mathbb{C}[K]$$

which sends $\partial_i \rightarrow [v_i]$. Since K_{prim} is finite, this kernel is a Noetherian module, thus a finite subset of (3) generates the rest.

Remark 2.3. The usual GKZ hypergeometric system coincides with $\text{GKZ}(\mathcal{A}, K; \beta)$ if N has no torsion and v_i generate K as a semigroup. Indeed, then $K_{\text{prim}} = \{0\}$, (2) leads to the linear equations of [GKZ] and (1) leads to

$$\left(\prod_{i=1}^n \partial_i^{k_i} \right) \Phi_0 = \left(\prod_{i=1}^n \partial_i^{l_i} \right) \Phi_0$$

whenever $\sum_i (k_i - l_i) v_i = 0$, which are the binomial relations of [GKZ].

Remark 2.4. The n -tuple \mathcal{A} of elements of N is allowed to contain repeated elements. As one can see from the PDEs defining the better-behaved GKZ system, the effect of having $v_i = v_j$ for some $i \neq j$, is that all functions Φ_c depend on $x_i + x_j$.

Example 2.5. Let $N = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $\mathcal{A} = (v_1, v_2)$, with $v_1 = (1, 0)$, $v_2 = (1, 1)$. Let β be an element in $N \otimes \mathbb{C} \cong \mathbb{C}$. The solution space of the better-behaved GKZ system is isomorphic to the space of pairs of functions $\Phi_{(0,0)}(x_1, x_2), \Phi_{(0,1)}(x_1, x_2)$ satisfying the equations

$$\partial_1 \Phi_{(0,0)} = \partial_2 \Phi_{(0,1)}, \quad \partial_2 \Phi_{(0,0)} = \partial_1 \Phi_{(0,1)},$$

$$(x_1 \partial_1 + x_2 \partial_2) \Phi_{(0,0)} = \beta \Phi_{(0,0)}, \quad (x_1 \partial_1 + x_2 \partial_2) \Phi_{(0,1)} = \beta \Phi_{(0,1)}.$$

The first pair of equations implies that both functions $\Phi_{(0,0)}$ and $\Phi_{(0,1)}$ satisfy the wave equation. It follows that

$$\Phi_{(0,0)}(x_1, x_2) = a(x_1 + x_2) + b(x_1 - x_2),$$

$$\Phi_{(0,1)}(x_1, x_2) = a(x_1 + x_2) - b(x_1 - x_2),$$

for some arbitrary functions a, b . The second pair of equations implies then that

$$(x_1 + x_2)a'(x_1 + x_2) = \beta a(x_1 + x_2), \quad (x_1 - x_2)b'(x_1 - x_2) = \beta b(x_1 - x_2).$$

It follows that $a(x_1 + x_2) = A(x_1 + x_2)^\beta$ and $b(x_1 - x_2) = B(x_1 - x_2)^\beta$, for some arbitrary complex constants A, B . Hence the better-behaved GKZ system has a two-dimensional solution space. Note that the discriminant locus of the system consists of the reducible curve $x_1^2 - x_2^2 = 0$ in \mathbb{C}^2 .

Definition 2.6. For any subset S of N which is closed under the additions of v_i we can define the system $\text{GKZ}(\mathcal{A}, S; \beta)$ as in Definition 2.1, but with $c \in S$ rather than $c \in K$.

Remark 2.7. If N has no torsion, then the usual version of GKZ is equivalent to $\text{GKZ}(\mathcal{A}, S; \beta)$ for S the subsemigroup of K generated by v_i . The fact that $\text{GKZ}(\mathcal{A}, K; \beta)$ is better-behaved than the usual GKZ is then related to the fact that the semigroup algebra $\mathbb{C}[K]$ is always Cohen-Macaulay, whereas $\mathbb{C}[S]$ need not be so.

3. SPACES OF SOLUTIONS OF THE BETTER-BEHAVED GKZ AND THE LOGARITHMIC JACOBIAN RING

Let $(x_1, \dots, x_n) \in \mathbb{C}^n$. We introduce a non-degeneracy notion for a degree one element $f = \sum_{i=1}^n x_i[v_i]$ of $\mathbb{C}[K]$ which is closely related to the one used by Batyrev [B] in the non-torsion case (see for example [B, Theorem 4.8]).

Definition 3.1. The degree one element $f = \sum_{i=1}^n x_i[v_i]$ of $\mathbb{C}[K]$ is said to be *non-degenerate* if the logarithmic derivatives $\sum_i x_i \mu_j(v_i)[v_i]$ form a regular sequence in $\mathbb{C}[K]$ for a basis μ_j , $1 \leq j \leq \text{rk} M$, of M .

Proposition 3.2. *For a generic choice of $f = \sum_{i=1}^n x_i[v_i]$ and any basis (μ_j) of M the log-derivatives $f_j = \sum_i x_i \mu_j(v_i)[v_i]$ of f give a regular sequence in $\mathbb{C}[K]$. Equivalently, the Koszul complex induced by the elements f_j*

$$(4) \quad 0 \rightarrow \dots \rightarrow \wedge^2 M \otimes \mathbb{C}[K] \rightarrow M \otimes \mathbb{C}[K] \rightarrow \mathbb{C}[K] \rightarrow R(f, K) \rightarrow 0$$

is exact.

Proof. If N has no torsion, the result is [Bo, Proposition 3.2]. The Koszul complex reformulation is standard. If N has torsion, the result appears to be new, but perhaps not particularly unexpected. In order to prove it, note that the ring $\mathbb{C}[K]$ is the direct sum of $|\text{tors} N|$ copies of $\mathbb{C}[\pi(K)]$, where $\pi : K \rightarrow K \otimes \mathbb{R}$ is the natural map. Then the

regularity of the sequence needs to be checked at each individual copy of $\mathbb{C}[\pi(K)]$ where it follows again from the non-torsion result. \square

Definition 3.3. The ring $R(f, K)$ is called the *logarithmic Jacobian ring* associated to f and K .

Corollary 3.4. *The dimension of the \mathbb{C} -vector space $R(f, K)$ is equal to $\text{vol}(\Delta) \cdot |\text{tors}(N)|$, where $\text{vol}(\Delta)$ is the normalized volume of the polytope Δ in $N \otimes \mathbb{R}$, and $|\text{tors}(N)|$ is the order of the torsion part of N .*

Proof. The dimension of the \mathbb{C} -vector space $R(f, K)$ is equal to the product of $(\text{rk} N - 1)! \cdot |\text{tors}(N)|$ and the leading coefficient of the Hilbert polynomial of the graded ring $\mathbb{C}[K \otimes \mathbb{Z}]$. But it is well known that this leading coefficient is the quotient of the normalized volume of Δ by $(\text{rk} N - 1)!$. \square

The complex (4) is graded with finite-dimensional graded components. We can dualize it component-wise to get another graded exact complex with finite-dimensional graded components

$$(5) \quad 0 \rightarrow R(f, K)^\vee \rightarrow \mathbb{C}[K] \rightarrow N \otimes \mathbb{C}[K] \rightarrow \wedge^2 N \otimes \mathbb{C}[K] \rightarrow \dots \rightarrow 0$$

We will naturally identify the (graded) dual of $\mathbb{C}[K]$ with itself, since each graded component of $\mathbb{C}[K]$ has a natural basis.

The complex (5) allows us to give the following description of the vector space $R(f, K)^\vee$.

Proposition 3.5. *The space $R(f, K)^\vee$ is the set of elements $\sum_{c \in K} \lambda_c [c]$ in $\mathbb{C}[K]$ such that the linear equations in $N \otimes \mathbb{C}$*

$$(6) \quad \sum_{i=1}^n x_i \lambda_{c+v_i} v_i = 0$$

hold for all $c \in K$.

Proof. The result follows from the observation that the dual of the map $M \otimes \mathbb{C}[K] \rightarrow \mathbb{C}[K]$ in the Koszul complex (4) is the map $\mathbb{C}[K] \rightarrow N \otimes \mathbb{C}[K]$ in the dual complex (5) given by

$$\sum_{c \in K} \lambda_c [c] \mapsto \sum_{c \in K} \sum_{i=1}^n x_i \lambda_{c+v_i} v_i \otimes [c].$$

\square

Remark 3.6. Note that equations (6) can be solved degree-by-degree and will have no nontrivial solutions for $\deg(c) > \text{rk} N$. Indeed, the exactness of the complex (5) implies that the Hilbert-Poincaré series of

the kernel of the map $\mathbb{C}[K] \rightarrow N \otimes \mathbb{C}[K]$ is a polynomial of degree at most $\text{rk} N$.

Let us now consider the solutions to $\text{GKZ}(\mathcal{A}, K; \beta)$.

Theorem 3.7. *The space of analytic solutions to $\text{GKZ}(\mathcal{A}, K; \beta)$ in a neighborhood of a generic f is isomorphic to the space of elements $\sum_{c \in K} \lambda_c [c]$ in $\mathbb{C}[K]^c$ such that the linear equations in $N \otimes \mathbb{C}$*

$$(7) \quad \sum_{i=1}^n x_i \lambda_{c+v_i} v_i = \lambda_c (\beta - c)$$

hold for all $c \in K$.

Proof. In one direction, if we have a solution $(\Phi_c), c \in K$, then $\lambda_c = \Phi_c(x_1, \dots, x_n)$ clearly satisfies (7). In fact, this map from the space of solutions of $\text{GKZ}(\mathcal{A}, K; \beta)$ to the space of solutions of (7) is clearly injective in view of Taylor's formula, since knowing all $\Phi_c(x_1, \dots, x_n)$ implies the knowledge of all the partial derivatives of all Φ_c at (x_1, \dots, x_n) in view of equation (1).

In the other direction, suppose that we have a solution (λ_c) of (7). Then equation (1) and Taylor formula force us to have

$$(8) \quad \Phi_c(z_1, \dots, z_n) = \sum_{(l_1, \dots, l_n) \in \mathbb{Z}_{\geq 0}^n} \lambda_{c + \sum_i l_i v_i} \prod_{i=1}^n \frac{(z_i - x_i)^{l_i}}{l_i!}$$

for all $c \in K$. It remains to show that the above series converges absolutely and uniformly in $c \in K$ and \mathbf{z} in a neighborhood of (x_1, \dots, x_n) . Observe that it suffices to show uniform convergence for a fixed $c \in K_{\text{prim}}$, since the partial derivative of a Taylor series will converge in the same neighborhood and $K_{\text{prim}} \subset K$ is a finite set. From now on we fix $c = c_0$.

We claim that there exists a constant $C_1 \in \mathbb{R}$ such that

$$(9) \quad |\lambda_{c_0 + \sum_i l_i v_i}| \leq C_1^{(\sum_{i=1}^n l_i)} \left(\sum_{i=1}^n l_i \right)!$$

for all nonzero $(l_1, \dots, l_n) \in \mathbb{Z}_{\geq 0}^n$. This is easily seen to be equivalent to the existence of a constant $C_2 \in \mathbb{R}$ such that

$$(10) \quad |\lambda_d| \leq C_2^{\deg d} (\deg d)!$$

for all d with sufficiently high $\deg d$.

Define $\Lambda_k = \max_{d, \deg d = k} |\lambda_d|$. To prove (10) it suffices to show that there exists $C_3 \in \mathbb{R}$ such that $\Lambda_{k+1} \leq C_3 k \Lambda_k$ for all sufficiently large k .

The ideal I of $\mathbb{C}[K]$ generated by logarithmic derivatives of f contains $[d]$ for all d of $\deg d = \text{rk}N + 1$. It is easy to see that every d_1 of sufficiently high degree can be written as $d_1 = d + d_2$ with $d, d_2 \in K$ and $\deg d = \text{rk}N + 1$. We can write each $[d]$ of degree $\text{rk}N + 1$ as

$$[d] = \sum_{i=1}^n \sum_{j=1}^{\text{rk}N} x_i \mu_j(v_i) [v_i] t_{d,j}$$

for some $t_{d,j} \in \mathbb{C}[K]_{\deg=\text{rk}N}$ and some basis $(\mu_1, \dots, \mu_{\text{rk}N})$ of M . Consequently, for each d_1 of sufficiently high degree we have that

$$[d_1] = \sum_{i=1}^n \sum_{j=1}^{\text{rk}N} x_i \mu_j(v_i) [v_i] t_{d,j} [d_2]$$

for some d_2 . By considering the maximum size of the coefficients of $t_{d,j}$ we observe that, for $\deg d_1 = k + 1$,

$$[d_1] = \sum_{i=1}^n \sum_{j=1}^{\text{rk}N} \sum_{d_3, \deg d_3 = k} \beta_{d_3,j} x_i \mu_j(v_i) [d_3 + v_i]$$

with $\sum_{d_3,j} |\beta_{d_3,j}|$ bounded by a constant independent of d_1 and k . Equation (7) implies then that

$$\sum_{d_3,j} \beta_{d_3,j} \lambda_{d_3} \mu_j(\beta - d_3) = \sum_{i,d_3,j} \beta_{d_3,j} x_i \lambda_{d_3+v_i} \mu_j(v_i) = \lambda_{d_1}.$$

Since $\sum_{d_3,j} |\beta_{d_3,j}|$ is bounded by a constant, $|\lambda_{d_3}|$ is bounded by Λ_k and $\mu(\beta - d_3)$ is bounded by a constant times k , we get $|\lambda_{d_1}| \leq C_3 k \Lambda_k$ as required.

This allows us to establish estimates (10) and (9). Since the multinomial coefficients

$$\frac{(\sum_{i=1}^n l_i)!}{\prod_{i=1}^n l_i!}$$

are bounded by $n^{\sum_{i=1}^n l_i}$, the terms of the series (8) are bounded by $\prod_{i=1}^n (nC_1)^{l_i} |z_i - x_i|^{l_i}$. By making $|z_i - x_i|$ sufficiently small, the absolute convergence is obtained by comparing to a product of convergent geometric series. \square

Having identified the space of solutions of $\text{GKZ}(\mathcal{A}, K; \beta)$ in a neighborhood of (x_1, \dots, x_n) with the space of solutions of the equations (7), we can now consider a natural filtration on it. We define the subspaces F_k of the space of solutions of $\text{GKZ}(\mathcal{A}, K; \beta)$ in a neighborhood of a generic (x_1, \dots, x_n) to be characterized by the fact that $\Phi_c(x_1, \dots, x_n) = 0$ for all c with $\deg c < k$. We have that $F_0 \supseteq F_1 \supseteq F_2 \supseteq \dots$.

Theorem 3.8. *The quotient F_k/F_{k+1} is naturally isomorphic to the dual of the degree k component of $R(f, K)$.*

Proof. The essential observation is that the equations (7) satisfied by the elements λ_c can be solved recursively in the degree of c . Indeed, suppose that we have found $\lambda_d, \deg d \leq k$, which satisfy (7) for all $c, \deg c \leq k-1$. In order to check that a solution exists for all d of degree $k+1$, we need to check that $\sum_{c, \deg c=k} \lambda_c(\beta - c)[c]$ sits in the degree k component of the image of the map $C[K]^c \rightarrow N \otimes \mathbb{C}[K]^c$ of the complex (5). Since this is an exact complex, it suffices to check that it is in the kernel of the map

$$N \otimes \mathbb{C}[K]^c \rightarrow \wedge^2 N \otimes \mathbb{C}[K]^c.$$

of (5). The coefficient of its image at $[c_1]$ is given by the element of $\wedge^2 N \otimes \mathbb{C}[K]^c$

$$\sum_{i=1}^n x_i \lambda_{c_1+v_i} v_i \wedge (\beta - c_1) = \lambda_{c_1}(\beta - c_1) \wedge (\beta - c_1) = 0.$$

Observe that as we are solving recursively the equations (7), the ambiguity at each step is precisely an element of the corresponding component of $R(f, K)^\vee$, which leads to the result. \square

Corollary 3.9. *The space of solutions to the true GKZ system is of the same dimension as $R(f, K)$.*

Remark 3.10. The same argument applies with obvious modifications when one replaces K by its interior K° .

Remark 3.11. The argument of this section is likely philosophically the same as the general arguments used in the theory of holonomic D -modules, but it has an advantage of being self-contained.

4. EFFECTS OF TORSION AND REPETITIONS

In this section we analyze the role of played by the torsion part of the finitely generated abelian group N and by the possible repetitions that may appear in the n -tuple \mathcal{A} . Let $\{w_1, \dots, w_m\} \subset N \otimes \mathbb{R}$ be the set consisting of the elements $\pi(v_i), 1 \leq i \leq n$, in $N \otimes \mathbb{R}$ where $\pi : N \rightarrow N \otimes \mathbb{R}$ is the natural map. For each $j, 1 \leq j \leq m$, let I_j be the set of indices i with $\pi(v_i) = w_j$.

Let $\rho : N \rightarrow \mathbb{C}^\times$ be a multiplicative group character. Define the map $p_\rho : \mathbb{C}^n \rightarrow \mathbb{C}^m$ by

$$(11) \quad p_\rho(x_1, \dots, x_n) := \left(\sum_{i \in I_1} \rho(v_i) x_i, \dots, \sum_{i \in I_m} \rho(v_i) x_i \right).$$

To a sequence of functions $(\Psi_w(z_1, \dots, z_m))_{w \in \pi(K)}$, we associate a sequence of functions $(\Phi_c(x_1, \dots, x_n))_{c \in K}$ such that, for any $c \in K$,

$$(12) \quad \Phi_c(x_1, \dots, x_n) := \rho(c) \Psi_{\pi(c)}(p_\rho(x_1, \dots, x_n)).$$

What makes this definition useful is the following result.

Proposition 4.1. *For any character $\rho \in \text{Hom}(N, \mathbb{C}^\times)$, if the function*

$$\Psi_{\pi(K)}(z_1, \dots, z_m) = \sum_{w \in p(K)} \Psi_w(z_1, \dots, z_m)[w],$$

is a solution on an open set $U \subset \mathbb{C}^m$ to the better behaved GKZ associated to $\beta \in N \otimes \mathbb{C}$ defined by $\{w_1, \dots, w_m\}$ in $\pi(N)$, then the associated function

$$\Phi_K(x_1, \dots, x_n) = \sum_{c \in K} \Phi_c(x_1, \dots, x_n)[c],$$

is a solution on the open set $p_\rho^{-1}(U) \subset \mathbb{C}^n$ to the better behaved GKZ associated to $\beta \in N \otimes \mathbb{C}$ defined by (v_1, \dots, v_n) in N .

Proof. For any $c \in N$, equation (12) implies that given some $v_i \in N$ and $w_j \in \pi(N)$ such that $\pi(v_i) = w_j$ we have that

$$\partial_i \Phi_c(x, \dots, x_n) = \rho(c + v_i) \partial_j \Psi_{\pi(c)} \left(\sum_{i \in I_1} \rho(v_i) x_i, \dots, \sum_{i \in I_m} \rho(v_i) x_i \right).$$

Since the functions $\Psi_w, w \in \pi(K)$, are solutions to the better behaved GKZ in $\pi(N)$, and $\pi(c) + w_j = \pi(c) + \pi(v_i) = \pi(c + v_i)$, we obtain indeed that $\partial_i \Phi_c(x, \dots, x_n) = \Phi_{c+v_i}$. Similarly, for any $\mu \in M = \text{Hom}(N, \mathbb{Z}) = \text{Hom}(\pi(N), \mathbb{Z})$, we have that

$$\begin{aligned} & \sum_{i=1}^n \mu(v_i) x_i \partial_i \Phi_c(x_1, \dots, x_n) \\ &= \rho(c) \sum_{j=1}^m \mu(w_j) \left(\sum_{i \in I_j} \rho(v_i) x_i \right) \partial_j \Psi_{\pi(c)} \left(\sum_{i \in I_1} \rho(v_i) x_i, \dots, \sum_{i \in I_m} \rho(v_i) x_i \right) \\ &= \mu(\beta - c) \Phi_c(x_1, \dots, x_n). \end{aligned}$$

□

Let G denote the torsion part of N . Since G is finite abelian group, we have that $\text{Hom}(G, \mathbb{C}^\times) \simeq G$. Assume that G has order k , and let $(\rho_g)_{g \in G}$ be the corresponding set of independent characters in $\text{Hom}(G, \mathbb{C}^\times) \simeq G$. When there is no torsion, we set $|G| = 1$ and $\rho_1 = 1$. The characters ρ_g can be easily extended to become multiplicative characters of N by imposing that they take the value 1 on the free part of N , after a choice of splitting. Under this convention, we will view the characters ρ_g as

elements in $\text{Hom}(N, \mathbb{C}^\times)$. As in formula (11), for each $g \in G$, we define the linear surjective maps $p_g : \mathbb{C}^n \rightarrow \mathbb{C}^m$ by

$$p_g(x_1, \dots, x_n) := \left(\sum_{i \in I_1} \rho_g(v_i) x_i, \dots, \sum_{i \in I_m} \rho_g(v_i) x_i \right).$$

Let $U \subset \mathbb{C}^m$ a nonempty open set in \mathbb{C}^m with the property that there exists an open set V in \mathbb{R}^m such that

$$(13) \quad U = \{(z_1, \dots, z_m) : (\log |z_1|, \dots, \log |z_m|) \in V, \\ (\arg z_1, \dots, \arg z_m) \in (-\pi, \pi) \times \dots \times (-\pi, \pi)\}$$

for a choice of the argument functions $(\arg z_1, \dots, \arg z_m) \in \mathbb{R}^m$. For such a set $U \subset \mathbb{C}^m$, we have that:

Lemma 4.2. $\bigcap_{g \in G} p_g^{-1}(U) \neq \emptyset$.

Proof. For each set of indices I_j , choose exactly one $i_j \in I_j$ and a complex number x_{i_j} such that

$$(\log |x_{i_1}|, \dots, \log |x_{i_m}|) \in V \setminus \{0\}$$

and, for all $j, 1 \leq j \leq m$,

$$\arg x_{i_j} + \pi < 2\pi/|G|.$$

From (13), we see that $\arg(\rho_g(v_{i_j})x_{i_j}) \in (-\pi, \pi)$, for any $g \in G$ and $1 \leq j \leq m$, hence

$$(\rho_g(v_{i_1})x_{i_1}, \dots, \rho_g(v_{i_m})x_{i_m}) \in U,$$

for any $g \in G$. By continuity, it is now possible to choose all the other complex numbers $x_i, 1 \leq i \leq n$, for those indices i different from any of the i_j 's, in a small enough neighborhood of the origin in the complex plane such that $p_g(x_1, \dots, x_n) \in U$, for any $g \in G$. The lemma follows. \square

Theorem 4.3. Let $\Psi_{\pi(K)}^\lambda, \lambda \in \Lambda$, be a set of linearly independent analytic solutions on an open set $U \subset \mathbb{C}^m$ satisfying condition (13) to the better behaved GKZ associated to $\beta \in N \otimes \mathbb{C}$ defined by (w_1, \dots, w_m) in $\pi(N)$. The associated set of $|\Lambda| \cdot |G|$ functions $\Phi_K^{\lambda, g}, \lambda \in \Lambda, g \in G$, is a set of linearly independent analytic solutions on the non-empty open set $p_1^{-1}(U) \cap \dots \cap p_m^{-1}(U) \subset \mathbb{C}^n$ to the better behaved GKZ associated to $\beta \in N \otimes \mathbb{C}$ defined by (v_1, \dots, v_n) in N .

Proof. Assume that there exists constants $\alpha_{\lambda, g}$ such that

$$\sum_{\lambda \in \Lambda, g \in G} \alpha_{\lambda, g} \Phi_c^{\lambda, g}(x) = 0,$$

for any $c \in K, x \in \cap_{g \in G} p_g^{-1}(U)$. It follows that

$$\sum_{\lambda \in \Lambda, g \in G} \alpha_{\lambda, g} \rho_g(c + c_h) \Psi_{\pi(c)}^\lambda(p_g(x)) = 0,$$

for any $c \in K, x \in \cap_{g \in G} p_g^{-1}(U)$, and $c_h \in K$ such that $\pi(c_h) = 0$.

For each fixed $c \in K$, we have $|G|$ linear relations indexed by $h \in G$. The orthogonality relations for the characters of the representations of the finite group G imply that

$$\sum_{\lambda \in \Lambda} \alpha_{\lambda, g} \Psi_{\pi(c)}^\lambda(p_g(x)) = 0,$$

for any $g \in G, c \in K$ and $x \in \cap_{g \in G} p_g^{-1}(U)$. The linear independence of the analytic functions $\Psi_{\pi(K)}^\lambda, \lambda \in \Lambda$, in U shows that $\alpha_{\lambda, g}$ are all zero. \square

5. THE BETTER BEHAVED GKZ SYSTEM WITH $\beta = 0$ AND THE RELATION TO THE MIXED HODGE STRUCTURES

In this section we study the *restriction map* from the space of solutions of $\text{GKZ}(\mathcal{A}, K; \beta)$ to that of $\text{GKZ}(\mathcal{A}, K^\circ; \beta)$ given by forgetting Φ_c for $c \in \partial K$. Our focus is primarily on the case $\beta = 0$ which differs significantly from the general case. Our results are closely related to (and can be in principle derived from) the work of Batyrev in [B] on mixed Hodge structures of hypersurfaces in algebraic tori but our treatment is much more direct and algebraic.

We start with elementary examples which show that dimension of the image of the restriction map depends on β .

Example 5.1. Let $K = \mathbb{Z}_{\geq 0}$ and $v_1 = 1$. Then the solution space to $\text{GKZ}(\mathcal{A}, K; \beta)$ is one-dimensional, generated by (Φ_k) , with $\Phi_k = (\frac{d}{dx_1})^k x_1^\beta$. In particular, when $\beta = 0$, the restriction to the solutions of $\text{GKZ}(\mathcal{A}, K^\circ; \beta)$ is zero.

Example 5.2. Consider $K \subseteq \mathbb{Z}^3$ generated by

$$v_1 = (0, 0, 1), v_2 = (0, 1, 1), v_3 = (1, 1, 1), v_4 = (1, 0, 1).$$

The solutions to $\text{GKZ}(\mathcal{A}, K; \beta)$ are determined uniquely by $\Phi_{(0,0,0)}$. When $\beta = 0$, they are given by $\Phi_{(0,0,0)}(x_1, \dots, x_4) = c_1 + c_2 \ln(\frac{x_1 x_3}{x_2 x_4})$ and thus restrict to zero solutions of $\text{GKZ}(\mathcal{A}, K^\circ; \beta)$.

Note that in the above examples the maps $R(f, K^\circ) \rightarrow R(f, K)$ are zero. The main result of this section is that the dimension of the image of the restriction map coincides with the dimension of the image of

$R(f, K^\circ) \rightarrow R(f, K)$ in the $\beta = 0$ case. Our approach uses the dual description of the solution space which we now introduce.

The semigroup ring $\mathbb{C}[K]$ is endowed with the structure of the module over $\text{Sym}^*(M_{\mathbb{C}})$ induced by the multiplication by the logarithmic derivatives of f . Specifically, the multiplication by the linear function $\mu : N \rightarrow \mathbb{C}$ is given by

$$(14) \quad \mu[n] = \sum_i x_i \mu(v_i)[n + v_i].$$

Consider a different module structure on $\mathbb{C}[K]$, where the multiplication is modified to be

$$(15) \quad \mu[\widehat{n}] = \sum_i x_i \mu(v_i)[\widehat{n + v_i}] + \mu(n - \beta)[\widehat{n}]$$

where we use the $\widehat{}$ notation to avoid confusion with (14). It is straightforward to check that $\mu_1 \mu_2 [\widehat{n}] = \mu_2 \mu_1 [\widehat{n}]$, so indeed $\mathbb{C}[K]$ is endowed with another module structure over the ring $\text{Sym}^*(M_{\mathbb{C}})$. This module structure is no longer compatible with the multiplication in $\mathbb{C}[K]$, rather the action is given by differential operators, see [B, Def. 7.2].

Definition 5.3. We will denote the space $\mathbb{C}[K]$ with the module structure given by (15) by $\widehat{\mathbb{C}[K]}$, where β should be clear from the context. The notation $\mathbb{C}[K]$ will imply the module structure of (14). The same convention will be used for other subsets of N , for example $\widehat{\mathbb{C}[K^\circ]}$ and $\mathbb{C}[K^\circ]$. We will also use $\widehat{[n]}$ to denote the basis elements of $\widehat{}$ spaces.

Remark 5.4. The $\widehat{\mathbb{C}[K]}$ module is *not* graded. However, it admits a filtration $(\widehat{\mathbb{C}[K]_{\leq k}})$ by the degree so that the associated graded module is naturally isomorphic to $\mathbb{C}[K]$.

Recall that $\mathbb{C}[K]$ is a free graded module over $\text{Sym}^*(M_{\mathbb{C}})$ for non-degenerate f , see [B, Theorem 4.8]. The following proposition shows that $\widehat{\mathbb{C}[K]}$ is also free.

Proposition 5.5. *The module $\widehat{\mathbb{C}[K]}$ is free for any β . Moreover, suppose we have a homogeneous basis $(u_i = \sum_n r_{n,i}[n])$ of $\mathbb{C}[K]$ as a free module over $\text{Sym}^*(M_{\mathbb{C}})$ and have chosen $w_i \in \widehat{\mathbb{C}[K]}$ made from monomials of degree lower than the degree of u_i . Then the elements $(\sum_n r_{n,i}[\widehat{n}] + w_i)$ freely generate $\widehat{\mathbb{C}[K]}$.*

Proof. If $(\sum_n r_{n,i}[\widehat{n}] + w_i)$ are linearly dependent over $\text{Sym}^*(M_{\mathbb{C}})$, consider the top degrees of the corresponding linear combination. Then we get a contradiction with the linear independence of $(\sum_n r_{n,i}[n])$. To

show that $(\sum_n r_{n,l} \widehat{[n]} + w_l)$ generate $\widehat{\mathbb{C}[K]}$ over $\text{Sym}^*(M_{\mathbb{C}})$, proceed by induction on degree. \square

Remark 5.6. It is easy to see that under the assumptions of Proposition 5.5, we get an isomorphism between $\mathbb{C}[K]$ and $\widehat{\mathbb{C}[K]}$, obtained by identifying the corresponding basis elements, which preserves the filtrations $\widehat{\mathbb{C}[K]}_{\leq \bullet}$ and $\mathbb{C}[K]_{\leq \bullet}$. In fact, it is an identity on the associated graded spaces (i.e. $\widehat{[n]} \mapsto [n] \bmod \mathbb{C}[K]_{\deg < \deg(n)}$) and any isomorphism with this property is described by the above proposition.

We will denote by I the irrelevant ideal of the graded ring $\text{Sym}^* M_{\mathbb{C}}$.

Corollary 5.7. *The image of $\widehat{\mathbb{C}[K]}_{\leq k}$ in $\widehat{\mathbb{C}[K]}/I\widehat{\mathbb{C}[K]}$ is naturally isomorphic to $\widehat{\mathbb{C}[K]}_{\leq k}/I\widehat{\mathbb{C}[K]}_{\leq k-1}$. The filtration on $\widehat{\mathbb{C}[K]}/I\widehat{\mathbb{C}[K]}$ induced from the filtration on $\widehat{\mathbb{C}[K]}$ has associated graded spaces naturally isomorphic to the graded components of $\mathbb{C}[K]/I\mathbb{C}[K]$.*

Proof. This follows immediately from Proposition 5.5. \square

Remark 5.8. Analogous statements regarding $\widehat{\mathbb{C}[K^\circ]}$ follow along the same lines.

The following result provides the link between the algebraic structures introduced in this section and the better behaved GKZ system.

Proposition 5.9. *Let f be nondegenerate. The spaces of analytic solutions to $\text{GKZ}(\mathcal{A}, K; \beta)$ and $\text{GKZ}(\mathcal{A}, K^\circ; \beta)$ in a small neighborhood of x are naturally isomorphic to the duals of the spaces $\widehat{\mathbb{C}[K]}/I\widehat{\mathbb{C}[K]}$ and $\widehat{\mathbb{C}[K^\circ]}/I\widehat{\mathbb{C}[K^\circ]}$ respectively.*

Proof. The statement follows immediately from Theorem 3.7 and Remark 3.10. \square

We will now focus our attention on the case $\beta = 0$. What's remarkable about this case is that the $\widehat{}$ structure induces similar structures on $\mathbb{C}[\theta]$, for all the faces θ of K . Specifically, we have the following.

Proposition 5.10. *Let $\beta = 0$. For a face θ of the cone K , consider the $\text{Sym}^*(M_{\mathbb{C}})$ module $\widehat{\mathbb{C}[\theta]}$ given by (15) for $v_i \in \theta$. This module structure descends to the module structure for $\text{Sym}^*((\mathbb{C}\theta)^\vee)$. This structure is also compatible with the natural surjection $\widehat{\mathbb{C}[K]} \rightarrow \widehat{\mathbb{C}[\theta]}$.*

Proof. The statements follow from the definitions. Note that $\beta = 0$ is essential. Otherwise, the module structure does not descend to $\text{Sym}^*((\mathbb{C}\theta)^\vee)$ because $\mu \in \text{Ann}(\theta)$ no longer act trivially on $\widehat{\mathbb{C}[\theta]}$. \square

The key result of this section is that among the various isomorphisms of Proposition 5.5 there exists an isomorphism of $\text{Sym}^*(M_{\mathbb{C}})$ modules $\widehat{\mathbb{C}[K]} \simeq \mathbb{C}[K]$ which is compatible with the restrictions to the faces.

Theorem 5.11. *Let $\beta = 0$. Then there exists (a non-canonical) collection of isomorphisms between $\text{Sym}^*((\mathbb{C}\theta)^\vee)$ modules $\widehat{\mathbb{C}[\theta]} \simeq \mathbb{C}[\theta]$ which commute with the surjections $\widehat{\mathbb{C}[\theta_1]} \rightarrow \widehat{\mathbb{C}[\theta_2]}$ and $\mathbb{C}[\theta_1] \rightarrow \mathbb{C}[\theta_2]$ for all face inclusions $\theta_2 \subseteq \theta_1$. Moreover, these isomorphisms may be chosen to preserve the filtration and to be identity on the associated graded spaces.*

Proof. Clearly, the statement of the theorem amounts to finding an isomorphism $\widehat{\mathbb{C}[K]} \simeq \mathbb{C}[K]$ of Proposition 5.5 and Remark 5.6 with the property that $\widehat{[n]}$ is mapped into $\sum_{n_1} \alpha_{n_1, n} [n]$ where $\alpha_{n_1, n}$ are only nonzero for n_1 in the interior of the smallest face of K that contains n . The isomorphism will be constructed by induction on $\theta \subseteq K$, inspired by the arguments of [BrL].

The base of induction $\theta = \{0\}$ is trivial. Suppose that the desired isomorphisms have been constructed for all $\sigma \subsetneq \theta$ for some face $\theta \subseteq K$, and that they are compatible with the face inclusions.

Consider the modules $\widehat{\mathbb{C}[\partial\theta]}$ and $\mathbb{C}[\partial\theta]$ over $\text{Sym}^*((\mathbb{C}\theta)^\vee)$ which are the cokernels of the inclusion maps $\widehat{\mathbb{C}[\theta^\circ]} \rightarrow \widehat{\mathbb{C}[\theta]}$ and $\mathbb{C}[\theta^\circ] \rightarrow \mathbb{C}[\theta]$ respectively. These modules admit “cellular” resolutions (cf. [BrL, Def. 3.3])

$$\begin{array}{ccccccc}
0 & \rightarrow & \widehat{\mathbb{C}[\partial\theta]} & \rightarrow & \bigoplus_{\substack{\sigma \subsetneq \theta \\ \text{codim}(\sigma, \theta)=1}} \widehat{\mathbb{C}[\sigma]} & \rightarrow & \bigoplus_{\substack{\sigma \subsetneq \theta \\ \text{codim}(\sigma, \theta)=2}} \widehat{\mathbb{C}[\sigma]} \rightarrow \dots \\
& & & & \downarrow & & \downarrow \\
0 & \rightarrow & \mathbb{C}[\partial\theta] & \rightarrow & \bigoplus_{\substack{\sigma \subsetneq \theta \\ \text{codim}(\sigma, \theta)=1}} \mathbb{C}[\sigma] & \rightarrow & \bigoplus_{\substack{\sigma \subsetneq \theta \\ \text{codim}(\sigma, \theta)=2}} \mathbb{C}[\sigma] \rightarrow \dots
\end{array}$$

with the vertical arrows being the isomorphisms constructed by the induction assumption. Thus there is a unique isomorphism $\widehat{\mathbb{C}[\partial\theta]} \rightarrow \mathbb{C}[\partial\theta]$ that makes the diagram commute. Moreover, this isomorphism will be compatible with the filtration and will be an identity on the associated graded objects, since the same is true for the vertical maps of the above diagram.

We will now show how to lift the isomorphism $\widehat{\mathbb{C}[\partial\theta]} \rightarrow \mathbb{C}[\partial\theta]$ to $\widehat{\mathbb{C}[\theta]} \rightarrow \mathbb{C}[\theta]$. Denote by I the irrelevant ideal of the polynomial ring

$\text{Sym}^*((\mathbb{C}\theta)^\vee)$. We have a surjection of graded vector spaces

$$\mathbb{C}[\theta]/IC[\theta] \rightarrow \mathbb{C}[\partial\theta]/IC[\partial\theta] \rightarrow 0.$$

Lift a graded basis S of $\mathbb{C}[\partial\theta]/IC[\partial\theta]$ to a set S_1 of graded elements of $\mathbb{C}[\theta]/IC[\theta]$. Then complete S_1 to a basis of $\mathbb{C}[\theta]/IC[\theta]$ by adding elements from a set S_2 which map to zero in $\mathbb{C}[\partial\theta]/IC[\partial\theta]$. We can lift the sets S_1 and S_2 to subsets of homogeneous elements S_3 and S_4 in $\mathbb{C}[\theta]$, respectively. We can do it in such a way as to ensure that S_4 maps to 0 in $\mathbb{C}[\partial\theta]$. Indeed, for any lift, the image is in $IC[\partial\theta]$, which is in the image of $IC[\theta]$. Thus the lifts can be adjusted to go to 0 by subtracting elements supported on the boundary of θ . We now define F_3 and F_4 as graded submodules of $\mathbb{C}[\theta]$ generated by S_3 and S_4 respectively. We have that $\mathbb{C}[\theta] = F_3 \oplus F_4$ by Nakayama's lemma where F_4 maps to zero in $\mathbb{C}[\partial\theta]$ and F_3 maps surjectively onto $\mathbb{C}[\partial\theta]$.

Let S_5 be the image of the set S_3 in $\mathbb{C}[\partial\theta]$. The set S_5 can be identified with a subset \widehat{S}_5 in $\widehat{\mathbb{C}[\partial\theta]}$ by the isomorphism we have already established. We then lift the set S_3 to a subset \widehat{S}_3 of $\widehat{\mathbb{C}[\theta]}$ so that the elements of \widehat{S}_3 have the same leading terms as those of S_3 . Similarly, we consider \widehat{S}_4 in $\widehat{\mathbb{C}[\theta]}$ which have the same leading terms as S_4 and restrict to zero on the boundary. Consider the morphism between $\mathbb{C}[\theta]$ and $\widehat{\mathbb{C}[\theta]}$ obtained by sending S_3 and S_4 to \widehat{S}_3 and \widehat{S}_4 respectively. It is an isomorphism by Proposition 5.5. It is also compatible with the isomorphism $\mathbb{C}[\partial\theta] \simeq \widehat{\mathbb{C}[\partial\theta]}$ by construction. This finishes the induction step. \square

For any β there is a natural map from the solutions of $\text{GKZ}(K, \beta)$ to the solutions of $\text{GKZ}(K^\circ, \beta)$ obtained by looking at $\Phi_c, c \in K^\circ$. Theorem 5.11 implies the following statement about the dimension of the image.

Corollary 5.12. *The space of solutions of $\text{GKZ}(K^\circ, 0)$ that can be extended to $\text{GKZ}(K, 0)$ has the same dimension as the image $R_1(f, K)$ of the map*

$$\mathbb{C}[K^\circ]/IC[K^\circ] \rightarrow \mathbb{C}[K]/IC[K]$$

considered in [BoM]. In fact, for a given point (x_i) this space admits a natural filtration by the order of vanishing at this point, and the associated graded space is naturally isomorphic to the dual of $R_1(f, K)$.

Proof. By Proposition 5.9, the dual of the restriction map in question is the map

$$\widehat{\mathbb{C}[K^\circ]}/\widehat{IC[K^\circ]} \rightarrow \widehat{\mathbb{C}[K]}/\widehat{IC[K]}$$

induced by the inclusion $\mathbb{C}[K^\circ] \subset \mathbb{C}[K]$.

By the argument of Theorem 5.11, the isomorphism constructed therein induces a commutative diagram

$$\begin{array}{ccccc} \widehat{\mathbb{C}[K]} & \rightarrow & \widehat{\mathbb{C}[\partial K]} & \rightarrow & 0 \\ \downarrow & & \downarrow & & \\ \mathbb{C}[K] & \rightarrow & \mathbb{C}[\partial K] & \rightarrow & 0 \end{array}$$

where vertical maps preserve the filtration and reduce to the identity on the associated graded spaces. By considering the kernels of the horizontal maps we get a commutative diagram of $\text{Sym}^*(M_{\mathbb{C}})$ modules

$$(16) \quad \begin{array}{ccccc} 0 & \rightarrow & \widehat{\mathbb{C}[K^\circ]} & \rightarrow & \widehat{\mathbb{C}[K]} \\ & & \downarrow & & \downarrow \\ 0 & \rightarrow & \mathbb{C}[K^\circ] & \rightarrow & \mathbb{C}[K] \end{array}$$

with the same property. This induces the commutative diagram

$$\begin{array}{ccc} \widehat{\mathbb{C}[K^\circ]}/I\widehat{\mathbb{C}[K^\circ]} & \rightarrow & \widehat{\mathbb{C}[K]}/I\widehat{\mathbb{C}[K]} \\ \downarrow & & \downarrow \\ \mathbb{C}[K^\circ]/I\mathbb{C}[K^\circ] & \rightarrow & \mathbb{C}[K]/I\mathbb{C}[K] \end{array}$$

which preserves the filtrations which are images of the corresponding filtrations before taking the quotient by I . We have thus shown that the images of the corresponding maps are isomorphic which implies the statement of the first sentence of this corollary.

We will now study the filtration and its associated graded object. The image of the part of the filtration that comes from $\widehat{\mathbb{C}[K^\circ]}_{\leq k}$ is given by

$$\widehat{\mathbb{C}[K^\circ]}_{\leq k}/(I\widehat{\mathbb{C}[K]}_{\leq k-1} \cap \widehat{\mathbb{C}[K^\circ]}_{\leq k}),$$

in view of the same statement for the image of $\widehat{\mathbb{C}[K^\circ]}_{\leq k}$. The associated graded piece is then given by

$$\widehat{\mathbb{C}[K^\circ]}_{\leq k}/(\widehat{\mathbb{C}[K^\circ]}_{\leq k-1} + (I\widehat{\mathbb{C}[K]}_{\leq k-1} \cap \widehat{\mathbb{C}[K^\circ]}_{\leq k}))$$

The isomorphism between this space and

$$\mathbb{C}[K^\circ]_{\leq k}/(\mathbb{C}[K^\circ]_{\leq k-1} + (I\mathbb{C}[K]_{\leq k-1} \cap \mathbb{C}[K^\circ]_{\leq k})) = R_1(f, K)_k$$

induced by Theorem 5.11 naturally lifts to the isomorphism between $\widehat{\mathbb{C}[K^\circ]}_{\leq k}/\widehat{\mathbb{C}[K^\circ]}_{\leq k-1}$ and $\mathbb{C}[K^\circ]_{\leq k}/\mathbb{C}[K^\circ]_{\leq k-1}$ which is natural in the sense that it does not depend on the choices of Theorem 5.11. This finishes the proof. \square

Remark 5.13. It is in general impossible to construct the commutative diagram (16) if $\beta \neq 0$. For instance, in the set up of the Example 5.1

we must have $\widehat{[0]} \mapsto [0]$ for the right vertical arrow. However, this isomorphism does not identify the ideals $\widehat{\mathbb{C}[K^\circ]}$ and $\mathbb{C}[K^\circ]$.

Remark 5.14. In place of $\mathbb{C}[K^\circ]$, we may similarly consider submodules $\widehat{\mathbb{C}[K]}$ which are linear combinations of monomials that are supported at faces of dimension at least as large as some given number. This gives a filtration on $\widehat{\mathbb{C}[K]}/I\widehat{\mathbb{C}[K]}$. Together with the filtration by the degree these are pretty much the weight and Hodge filtration on the cohomology of a hypersurface in algebraic torus, see [B].

Remark 5.15. The result of Corollary 5.12 can be presumably deduced from [B, section 8], but we were unable to find a precise reference. At any rate, this would have been a very roundabout derivation, as compared to our method based on Theorem 5.11.

6. FURTHER COMMENTS AND OPEN PROBLEMS

One can consider a *partial semigroup* version of the better-behaved GKZ hypergeometric system as follows. Let Σ be a fan in the cone K spanned by v_i . We can replace the equations $\partial_i \Phi_c = \Phi_{c+v_i}$ in Definition 2.1 by

$$\partial_i \Phi_c = \begin{cases} \Phi_{c+v_i}, & \text{there exists a cone of } \Sigma \text{ that contains } v_i \text{ and } c \\ 0, & \text{otherwise.} \end{cases}$$

Then if Σ admits a piecewise linear strictly convex function, then all the arguments of this paper are applicable to this more general system.

The properties of the better behaved version of the GKZ system show that this system is better suited than the usual GKZ if any type of functorial considerations are to be invoked. We expect that the mirror symmetric identification of the Fourier-Mukai transform and the analytic continuation formulae for the solutions to the GKZ system discussed in [BoH] continues to hold in the better behaved GKZ case. In fact, the context of this paper is more natural since the rank of the space of local solutions to the better behaved GKZ one side matches the rank of the orbifold cohomology/stacky K -theory on the other side. This was not the case in our previous work. It would be an interesting problem to study an appropriately defined category of better behaved GKZ systems and its functorial properties, part of which would mirror the properties of category of toric DM stacks. In the same realm, we expect that, in an appropriate sense, the system $\text{GKZ}(\mathcal{A}, K; \beta)$ is dual to the system $\text{GKZ}(\mathcal{A}, K^\circ; -\beta)$.

More importantly, we believe that the better behaved GKZ system lends itself to a process of categorification, which as a first step, is expected to provide a non-commutative categorical resolution of a Gorenstein toric singularity. Such a categorification will have to pass all the toric mirror symmetric checks, and, as such, would have a transcendental component which is missing in the algebraic proposals of non-commutative resolutions currently available in the literature. We hope to come back to this problem in a future paper.

REFERENCES

- [A] A. Adolphson, *Hypergeometric functions and rings generated by monomials*, Duke Math. J. 73 (1994), no. 2, 269–290.
- [A1] A. Adolphson, *Letter to P. Horja*, unpublished.
- [B] V. V. Batyrev, *Variations of mixed Hodge structure of affine hypersurfaces in algebraic tori*, Duke Math. J. 69 (1993), 349–409.
- [BvS] V. Batyrev, D. van Straten, *Generalized hypergeometric functions and rational curves on Calabi-Yau complete intersections in toric varieties*, Comm. Math. Phys. 168 (1995), no. 3, 493–533.
- [Bo] L. A. Borisov, *String cohomology of a toroidal singularity*, J. Algebraic Geom. 9 (2000), no. 2, 289–300, [math.AG/9802052](#).
- [BoCS] L. A. Borisov, L. Chen, G. Smith, *The orbifold Chow ring of toric Deligne-Mumford stacks*, J. Amer. Math. Soc. 18 (2005), no. 1, 193–215, [math.AG/0309229](#).
- [BoH] L. A. Borisov, R. P. Horja, *Mellin-Barnes integrals as Fourier-Mukai transforms*, Adv. Math. 207 (2006), no. 2, 876–927, [math.AG/0510486](#).
- [BoM] L. A. Borisov, A. R. Mavlyutov, *String cohomology of Calabi-Yau hypersurfaces via mirror symmetry*, Adv. Math. 180 (2003), no. 1, 355–390, [math.AG/0109096](#).
- [BrL] P. Bressler, V. Lunts, *Intersection cohomology on nonrational polytopes*, Compositio Math. 135 (2003), no. 3, 245–278.
- [CDS] E. Cattani, A. Dickenstein, B. Sturmfels, *Rational hypergeometric functions*, Compositio Math. 128 (2001), no. 2, 217–239, [math.AG/9911030](#).
- [GKZ] I. M. Gelfand, M. M. Kapranov, A. V. Zelevinsky, *Hypergeometric functions and toric varieties*, (Russian) Funktsional. Anal. i Prilozhen. 23 (1989), no. 2, 12–26; translation in Funct. Anal. Appl. 23 (1989), no. 2, 94–106.
- [HLY] S. Hosono, B.H. Lian, S.-T. Yau, *Maximal degeneracy points of GKZ systems*, J. Amer. Math. Soc. 10 (1997), no. 2, 427–443, [alg-geom/9603014](#).
- [I] H. Iritani, *Quantum Cohomology and Periods*, preprint (2011), <http://arxiv.org/abs/1101.4512>.
- [MMW] L. F. Matusevich, E. Miller, U. Walther, *Homological methods for hypergeometric families*, J. Amer. Math. Soc. 18 (2005), no. 4, 919–941, [math.AG/0406383](#).
- [SST] M. Saito, B. Sturmfels, N. Takayama, *Gröbner deformations of hypergeometric differential equations*, Springer, 2000.
- [S] J. Stienstra, *Resonant hypergeometric systems and mirror symmetry*, Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), 412–452, World Sci. Publishing, River Edge, NJ, 1998, [math.AG/9711002](#).

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